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Shock-like structure of phase-change flow in porous media

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Shock-like features of phase-change flows in porous media are explained, based on the generalized Darcy model. The flow field consists of two-phase zones of parabolic/hyperbolic type as well as adjacent or imbedded single-phase zones of either parabolic (superheated, compressible vapour) or elliptic (subcooled, incompressible liquid) type. Within the two-phase zones or at the two-phase/single-phase interfaces, there may be steep gradients in saturation and temperature approaching shock-like behaviour when the dissipative effects of capillarity and heat-conduction are negligible. Illustrative of these shocked, multizone flow-structures are the transient condensing flows in porous media, for which a self-similar, shock-preserving (Rankine–Hugoniot) analysis is presented.

1. Introduction

Geological applications motivate the study of transient phase-change flow in porous media. Examples include: geothermal systems (Brownell, Garg & Pritchett 1977), steam stimulation of oil fields (Weinstein, Wheeler & Woods 1977), and containment of underground nuclear tests (Morrison 1973) as well as the *in situ* combustion processes such as oil-shale retorting and coal gasification.

A mathematical statement of the conservation principles leads to partial differential equations having hyperbolic, parabolic and elliptic character within different regions of the flow. In phase-change regions, where the fluid-matrix energy transfer predominates, the transport equations are of a mixed parabolic/hyperbolic type. In adjacent or imbedded single-phase regions, the velocity field becomes nearly uncoupled from the temperature field and the pressure field is either parabolic or elliptic for the respective cases of compressible vapour and incompressible liquid. Transitions between zones are accompanied by steep gradients in saturation and temperature, approaching shock-like behaviour as capillary pressure and thermal conduction become negligible.

Saturation shock is a characteristic feature of multiphase flows in which the pressure gradient is the primary driving force rather than the gradient in capillary pressure, as expected in the applications noted above (although not in unsaturated hydrology, the infiltration problem, or some drying processes). The best known example of saturation shock is the Buckley–Leverett case of immiscible fluid/fluid displacement (Bear 1970). Comparable behaviour occurs in isothermal phase-change

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systems, as reviewed by Nikolaevskii & Somov (1978), but here the isothermal restriction precludes the fluid/matrix energy transfer which is paramount in the applications noted above. When the energy transfer is included as in oil displacement by hot water (Fayers 1962), thermal shocks are found to accompany the saturation shocks, provided that convective heat transfer dominates over conduction. These fundamental examples suggest that a composite of shock-like behaviour will likely be encountered in the coupled problem of non-isothermal, phase-change flow. Although it is true that capillarity and heat conduction will always smear the shock fronts in direct analogy with viscous smearing of gasdynamic shocks (Scheidegger 1974), these dissipative effects should be moderate in the noted applications, as is already apparent in some previous numerical simulations.

Shock-like phenomena are observed in numerical simulations of non-isothermal, phase-change flows in porous media (e.g. Weinstein *et al.* 1977; Morrison 1973), but there have been no analyses which explain the mathematical and physical character of these phase-change shocks which occur as a consequence of fluid/matrix energy transfer. Such an analysis is particularly needed because the direct numerical integration of the primitive equations is a very difficult task (subject to the numerical instabilities and dispersion which result from nonlinearity, type-change, and sharp fronts (Settari & Aziz 1975)). There has been no opportunity to assess the accuracy by comparison with a reliable but non-trivial solution, and the physical structure of the flow has been obscured by numerical smearing.

In the present study of phase-change shocks, consideration is given to self-similar flows. The ordinary differential equations are solved by a shock-preserving method, using Rankine–Hugoniot (jump balance) conditions in crossing the shock fronts. A representative example problem is that of transient condensing flow of a pure substance within a porous matrix. Depending on the initial and boundary conditions, several flow structures are found to occur as described in the individual sections of the paper:

(*a*) two-phase flow divided by a saturation shock (§ 4);

(*b*) two-phase flow divided by an imbedded slug of subcooled liquid, with shocks on both sides of the slug (§ 5);

(*c*) superheated inflow shocking into a two-phase zone like either (*a*) or (*b*) above (§ 6),

(*d*) two-phase inflow shocking into a superheated vapour zone, followed by a two-phase zone like either (*a*) or (*b*) above (§ 7);

(*e*) entry flow like either (*c*) or (*d*), shocking into a central two-phase zone, followed by a fully-wet subcooled far-field flow (§ 5).

Thus, the central structure is generally two-phase, divided by either a shock or an imbedded slug of liquid. The inner and outer zones respectively depend upon the boundary (inflow) data and the initial (far-field) data.

The primary purpose is to communicate the structure of the flow, based on a widely-used mathematical description of the physics. To accent the shock-like structure, the dispersive effects of capillarity and heat conduction are suppressed. The shock-preserving, self-similar method of solution is well suited because it affords the opportunity for rigorous analysis as well as reliable numerical computation based on well-established algorithms for ordinary differential equations. Qualitative observations and structural aspects are representative of a broad class of flows, not just the considered self-similar examples.

2. Transport equations

The transient, two-phase flow of a pure substance in a porous medium is governed by conservation of mass, energy and momentum (Cheng 1978; Whitaker 1977):

$$\frac{\partial}{\partial t}[\epsilon S\rho_l+\epsilon(1-S)\rho_v]+\frac{\partial}{\partial x}[\rho_l u_l+\rho_v u_v]=0; \tag{1a}$$

$$\frac{\partial}{\partial t}[\epsilon S\rho_l h_l+\epsilon(1-S)\rho_v h_v+(1-\epsilon)\rho_m h_m] \\ +\frac{\partial}{\partial x}[\rho_l h_l u_l+\rho_v h_v u_v]-\frac{\partial}{\partial x}\left[\langle k\rangle\frac{\partial T}{\partial x}\right]-\frac{DP}{Dt}=0; \tag{1b}$$

$$\left.\begin{aligned} u_v &= -\alpha_v\frac{K}{\mu_v}\frac{\partial P}{\partial x}, \quad \alpha_v = 1-S;\\ u_l &= -\alpha_l\frac{K}{\mu_l}\frac{\partial P}{\partial x}, \quad \alpha_l = S^3, \end{aligned}\right\} \tag{1c}$$

where the subscripts l, v, and m refer to liquid, vapour and solid matrix; K and ϵ are permeability and porosity; and S is the volume fraction of the pore space containing liquid. All other variables have the usual meaning. In the generalized Darcy equations, which relate velocity to pressure gradient at low Reynolds number, the relative permeability functions α_l and α_v are taken in a simple form which facilitates the analysis while still representing the proper qualitative behaviour (Scheidegger 1974; Wooding & Morel-Seytoux 1976). Although experimentally-determined α_l and α_v are considerably more complex, particularly near the single-phase extremes at $S = 0$ and $S = 1$, the basic qualitative behaviour of the flow should be essentially the same for any smooth monotonic functions (as verified by obtaining some comparative solutions in which both a_l and α_v were presumed linear in S).

Body forces and capillary pressure are neglected, thermal equilibrium between fluid and solid is presumed, and under the supposition of a high Peclet number, the conduction terms need only be included for the discussion of shock structure. Viscosities are assumed constant, the liquid is incompressible, the gas is ideal ($\rho = P/\tilde{R}T$), and the enthalpies $h_i = e_i+P/\rho_i$ depend linearly on T with slope C_i for $i = l, v, m$. Consistent with the low-Reynolds-number Darcy approximation, the kinetic energy and $\mathbf{u}\,.\,\nabla P$ work terms are neglected.

In a region of two-phase flow, the pressure and temperature are related by the Clausius–Clapeyron equation

$$\frac{dP}{dT}=\frac{h_{lv}}{Tv_{lv}}, \quad T = T_{\text{sat}}(P) \tag{2a}$$

in which $h_{lv} = h_v-h_l > 0$ and $v_{lv} = \rho_v^{-1}-\rho_l^{-1} > 0$. In a single-phase region, it is instead required that

$$\begin{aligned} S &= 1, \quad T < T_{\text{sat}}(P),\\ S &= 0, \quad T > T_{\text{sat}}(P), \end{aligned} \tag{2b}$$

for the cases of subcooled liquid and superheated vapour, respectively.

The initial and boundary conditions to be imposed are

$$S(x,0) = S_\infty, \quad P(x,0) = P_\infty, \quad T(x,0) = T_\infty;$$
$$S(0,t) = S_0, \quad P(0,t) = P_0, \quad T(0,t) = T_0. \tag{3}$$

To induce a forward flow ($\partial P/\partial x < 0$) and vapour condensation, the boundary pressure and temperature are abruptly increased to $P_0 > P_\infty$, $T_0 > T_\infty$. If the driving state is saturated, $T_0 = T_{\text{sat}}(P_0)$, and S_0 must be specified. If superheated, T_0 and P_0 are independent, but S_0 must vanish. In either case, there are two independent boundary conditions at $x = 0$; and similarly, there are two independent initial conditions.

The system reduces to a set of ordinary differential equations under the similarity transformation (Morrison 1973; Nikolaevskii & Somov 1978)

$$\theta = \frac{x}{t^{\frac{1}{2}}}\left(\frac{\epsilon\mu_v}{P_0 K}\right)^{\frac{1}{2}}.$$

Normalizing P, T, ρ, h_{lv}, C_i and k by P_0, T_0, ρ_{v0}, h_{lv0}, h_{lv0}/T_0 and k_0, respectively, the transformed equations are

$$\tfrac{1}{2}\theta(\rho_l S + (1-S)\rho_v)' + ((\rho_v\alpha_v + R\rho_l\alpha_l)P')' = 0, \tag{4a}$$

$$T'(F_h) - h_{lv}(\tfrac{1}{2}\theta\rho_l S' + R\rho_l(\alpha_l P')') - \tfrac{1}{2}\theta\Gamma P' + Pe^{-1}(kT')' = 0, \quad \rho_v = P/T, \tag{4b,c}$$

in which the derivatives, denoted $(\ )'$, are taken with respect to the similarity variable θ, and the parameters

$$R = \mu_v/\mu_l, \quad \Gamma = \tilde{R}T_0/h_{lv0}, \quad Pe^{-1} = k_0 T_0 \mu_v \epsilon/\rho_0 h_{lv0} K P_0$$

are all small numbers.† The convective energy flux involves the group

$$F_h = C_l F_l + C_v F_v + \rho_m C_m(\tfrac{1}{2}\theta)(1-\epsilon)/\epsilon$$

in which we introduce the notation

$$F_l = \rho_l(\tfrac{1}{2}\theta S + R\alpha_l P'), \quad F_v = \rho_v\alpha_v(\tfrac{1}{2}\theta + P')$$

for the mass flux of liquid and vapour relative to the moving self-similar co-ordinate system. The transformed boundary conditions are

$$S(0) = S_0, \quad P(0) = 1, \quad T(0) = 1;$$
$$S(\infty) = S_\infty, \quad P(\infty) = P_\infty, \quad T(\infty) = T_\infty, \tag{5}$$

with $P_\infty < 1$ and $T_\infty < 1$.

Within a two-phase zone, the equations are conveniently written

$$\mathbf{A}\begin{pmatrix} P'' \\ S' \end{pmatrix} = \mathbf{b}, \tag{6a}$$

$$\mathbf{A} = h_{lv}\begin{pmatrix} \rho_v\alpha_v & -\rho_v(\tfrac{1}{2}\theta + P') \\ \rho_l\alpha_l R & \rho_l(\tfrac{1}{2}\theta + 3RS^2P') \end{pmatrix}, \tag{6b}$$

$$\mathbf{b} = \begin{pmatrix} -h_{lv}\rho_v'\alpha_v(\tfrac{1}{2}\theta + P') - T'F_h \\ T'F_h \end{pmatrix}, \tag{6c}$$

$$\det\mathbf{A} = h_{lv}^2\rho_l\rho_v(\alpha_v(\tfrac{1}{2}\theta + 3RS^2P') + R\alpha_l(P' + \tfrac{1}{2}\theta)). \tag{7}$$

† From now on, both Γ and Pe^{-1} will be neglected, except for the discussion of shock structure in §6.

In single-phase regions, the energy equation (4*b*) reduces to

$$T'(F_h) = 0 \tag{8}$$

(in which F_h is somewhat degenerate since either F_v or F_l vanishes in one-phase regions), and the continuity equation (4*a*) reduces to either

$$P'' = 0, \tag{9a}$$

or

$$(\rho_v(P'+\tfrac{1}{2}\theta))' = \tfrac{1}{2}\theta\rho_v, \tag{9b}$$

for the liquid and vapour cases, respectively.

Although the system is third order, there are four independent boundary conditions suggesting that added flexibility is needed. It is noted that for $S_0 > 0$, $\det \mathbf{A} < 0$ at the origin, but that $\det \mathbf{A} \to +\infty$ as $\theta \to \infty$. Either the flow contains a singularity at which $\det \mathbf{A} = 0$ or a shock at which $\det \mathbf{A}$ changes sign. The first alternative affords the needed flexibility only if $\mathbf{b}$ becomes orthogonal to all solutions of $\mathbf{A}^T y = 0$ whenever $\det \mathbf{A} = 0$ – this being a sufficient condition for the existence of a singular subinterval of variable breadth. Since this compatibility condition is not automatically satisfied, a shock must be present.

3. Shock conditions

Mass, energy and momentum must be conserved in crossing a shock. From this fact (or by integrating (4) across a shock), we obtain the following shock conditions (Slattery 1972):

$$[F_l]+[F_v] = 0, \tag{10a}$$

$$[T]F_h + \hat{h}_{lv}[F_v] = 0, \tag{10b}$$

$$[P] = 0, \tag{10c}$$

in which the circumflex on h_{lv} indicates that it is to be evaluated on a different side of the shock than the quantities in F_h. Since h_{lv} may be evaluated on either side, and since h_{lv} has the same (positive) sign on both sides, so must F_h have the same sign on both sides. The pressure cannot jump (in 10*c*) because the Reynolds number is presumed low in Darcy flow and the inertial terms are, therefore, absent.

The entropy cannot decrease in crossing a shock. Letting ϕ denote the specific entropy, this condition can be written

$$[\phi_l F_l + \phi_v F_v + \phi_m \rho_m (\tfrac{1}{2}\theta)(1-\epsilon)/\epsilon] \leqslant 0. \tag{11}$$

Since the pressure does not change in crossing shocks

$$d\phi_i = \frac{dh}{T} = \frac{C_i dT}{T} \Rightarrow [\phi_i] = C_i[\ln T]; \quad i = v, l, m.$$

Using this result and the identity $\hat{\phi}_{lv} = \hat{h}_{lv}/T$, the second law (11) is combined with the energy equation (10*b*) to arrive at the inequality

$$\hat{T}\left([\ln T] - \frac{1}{\hat{T}}[T]\right) F_h \leqslant 0. \tag{12}$$

in which quantities with circumflexes lie on the two-phase side. To examine the consequences of this statement, first note that

$$[\ln T] - \frac{1}{\hat{T}}[T] \geqslant 0$$

whenever the two-phase region is on the right, and that this inequality changes direction whenever the two-phase region is on the left. It is, therefore, concluded that only three possibilities are consistent with the second law:

$$[T] = 0, \tag{13a}$$

$$[T] \neq 0, \text{ the two-phase region is on the left, and } F_h \geqslant 0, \tag{13b}$$

$$[T] \neq 0, \text{ the two-phase region is on the right, and } F_h \leqslant 0. \tag{13c}$$

4. Two-phase/two-phase

Consider the simplest case of a strictly two-phase flow (containing no single-phase regions) as occurs whenever $N = P_1/P_0$ and $S(\infty)$ are not too large and $S(0)$ is not too small. As already mentioned, there must be a shock at which $\det \mathbf{A}$ jumps from negative to positive. But, in passing from a two-phase region into another two-phase region, $[P] = 0 \Rightarrow [T] = 0$, so that the jump conditions $(10a, b)$ reduce to

$$[F_l] = [F_v] = 0$$

or, equivalently,

$$[S](\tfrac{1}{2}\theta + RP'(S^2 + \hat{S}^2 + S\hat{S})) + R\hat{S}^3[P'] = 0 \tag{14a}$$

$$(1 - \hat{S})[P'] = [S](\tfrac{1}{2}\theta + P') \tag{14b}$$

where quantities with circumflexes will now represent quantities on the right side of the shock.

The two-phase shock conditions $(14a, b)$ combine to give a cubic equation for $\hat{S}$

$$F(\hat{S}) = (\tfrac{1}{2}\theta + RP'(S^2 + S\hat{S} + \hat{S}^2))(1 - \hat{S}) + R\hat{S}^3(\tfrac{1}{2}\theta + P') = 0. \tag{15}$$

The existence of a unique physical solution is demonstrated by examining the behaviour of $F(\hat{S})$.

$$F(-\infty) = -\infty, \quad F(0) = \tfrac{1}{2}\theta + RP'S^2 \geqslant 0, \tag{16a, b}$$

$$F(S) = \frac{\det \mathbf{A}}{\rho_l \rho_v h_{lv}^2} < 0, \quad F(1) = R(\tfrac{1}{2}\theta + P') \leqslant 0, \quad F(\infty) = +\infty. \tag{16c, d, e}$$

The above inequalities on $F(1)$ and $F(0)$ are based on the observation that the velocities of vapour and liquid, each measured with respect to the shock,

$$V_v = -(\tfrac{1}{2}\theta + P') \quad \text{and} \quad V_l = -(\tfrac{1}{2}\theta + RP'S^2) \tag{17}$$

must have positive and negative signs, respectively, in order that it be possible for $\det \mathbf{A}$ to have the necessary sign-change in crossing. (The supportive argument is based on the following observations: $\det \mathbf{A}$ is roughly a linear combination of V_v and V_l, V_v and V_l have the same sign at the origin, V_v and V_l cannot both change sign to the left of the shock without a singularity, neither V_v nor V_l can change sign at the shock, and $V_v > V_l$.) The sign changes in (16) show that there are three real roots to the cubic equation $F(\hat{S}) = 0$.

$$\hat{S}_1 < 0, \quad \hat{S}_2 > 1, \quad 0 < \hat{S}_3 < S.$$

Since only $\hat{S}_3$ is physically meaningful, it is concluded that:

(*a*) a unique solution exists;

(*b*) $[S] < 0$, so the shock faces forward;

(*c*) $[P'] > 0$, from (14*b*);

(*d*) $\det \hat{\mathbf{A}} = R\hat{P}'\alpha_v(S+2\hat{S})[S]+[P'](\hat{S}^2+S^2+S\hat{S})\alpha_v+R\alpha_l > 0$, from (15).

The last inequality guarantees that $\det \mathbf{A}$ has the necessary sign-change in crossing the shock.

The stability of the shock can be assessed from the local features noted above. The inequalities (17) on V_v and V_l are sufficient to establish the one-dimensional stability in the sense that the characteristics on the left are overtaking the characteristics on the right. In addition, the inequality on $[P']$ is, according to the steady planar analyses of Miller (1975) and of Laude & Morrison (1979), sufficient to suggest the stability of the present flows under two-dimensional perturbations. In some of the more complex flow structures to be discussed in later sections it is not so easy to determine the stability from an *a priori* analysis, but an examination of the computed results (particularly the sign of $[P']$) suggests that the criterion is satisfied.

Numerical solutions are obtained by a forward-marching shooting method, as described in the appendix. The shooting parameters are $P'(0)$ and θ_s. The ordinary differential equations (6) are integrated outward to θ_s; the jump conditions (14*a*, *b*) are used to cross the shock; integration of (6) is resumed. The values of the shooting parameters are adjusted until the far-field boundary conditions are satisfied.

Typical profiles of S, P, and T are shown in figure 1. Upon increasing $S(0)$, as in figure 2, the saturation profile becomes spike-shaped at the leading edge. For large enough $S(0)$, say $S^*(0)$, the peak of the spike rises to $S = 1$, indicating liquid-full conditions behind the shock. The algorithm still converges for $S(0) > S^*(0)$, but the answers are unphysical since $S > 1$ in the region immediately behind the shock. Thus, for $S(0) \geqslant S^*(0)$ we seek to accommodate the excess liquid by making allowance for a liquid-full zone of finite width, as described in the next section.

5. Two-phase/subcooled liquid/two-phase

A condensing flow may contain a subcooled-liquid zone which lies imbedded within an otherwise two-phase region (figure 3). Such a situation arises as the continuous extension of a strictly two-phase flow under a change of data which favours liquid-flooding of the pore space: increase of $S(0)$ as in figure 2, increase of $S(\infty)$, increase of $\Delta T = T_0 - T_\infty$. In the transition from a strictly two-phase flow to an imbedded-liquid flow, the shock-line of the two-phase flow broadens into a liquid-filled zone of finite width.

Within the subcooled zone the structure is simple: $S = 1$, $T' \approx 0$ from (8) with $\Gamma \ll 1$, and $P' =$ constant for an incompressible liquid. The last implies uniformity of the liquid velocity as in a so-called slug-flow. Aside from these consequences of the

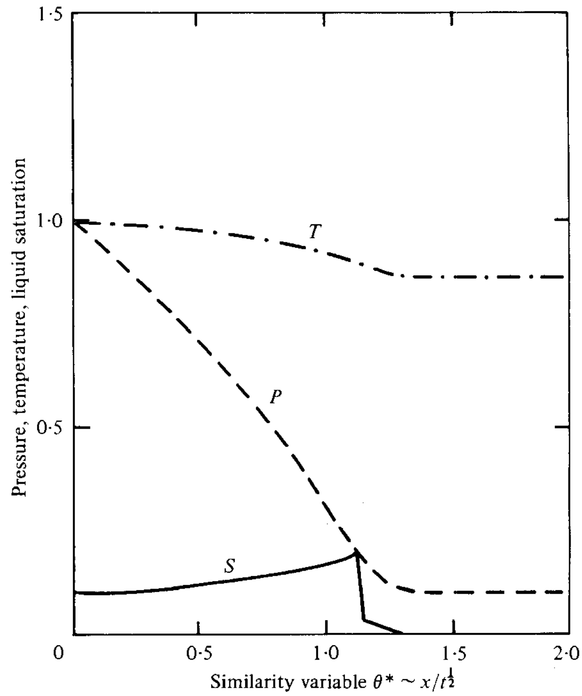

FIGURE 1. Strictly two-phase flow divided by a saturation jump.

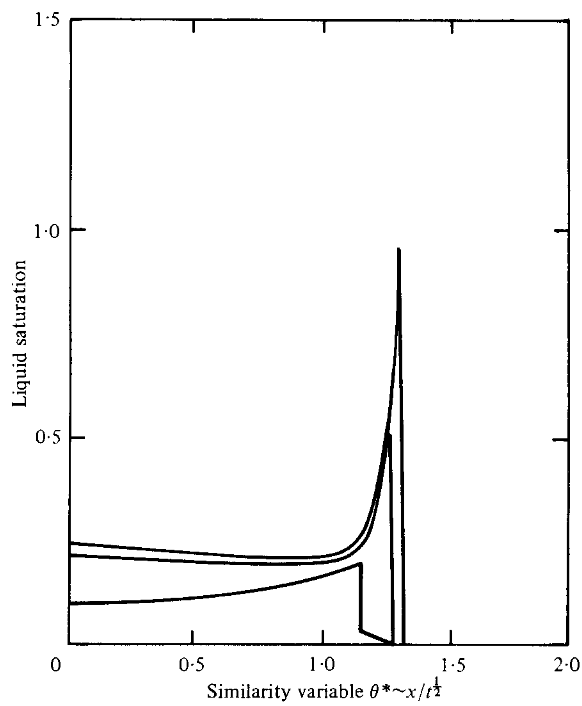

FIGURE 2. Family of saturation profiles for different prescriptions of the inflow saturation $S(0)$.

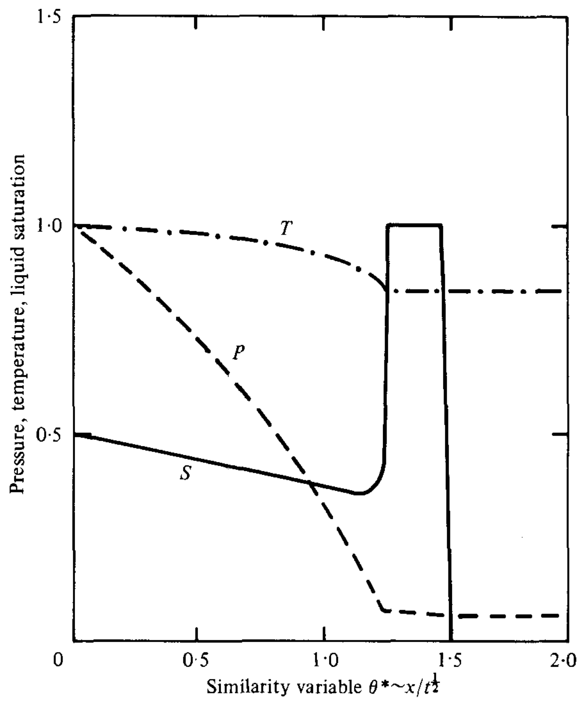

FIGURE 3. Two-phase flow divided by imbedded slug of subcooled liquid.

conservation equations, there is the thermodynamic requirement that $T \leqslant T_{sat}(P)$ everywhere within the subcooled slug.

A temperature jump $[T] < 0$ must occur at the left end of the slug $\theta = \theta_s$. Recall that the fluid temperature T and the saturation temperature $T_{sat}(P)$ are identical at θ_{s-}. Now, in crossing the slug, the saturation temperature must decrease (since $P' < 0$), while the fluid temperature remains nearly uniform (from (8))

$$\frac{dT_{sat}}{d\theta} < 0 \quad \text{and} \quad \frac{dT}{d\theta} = 0.$$

Were it not for an abrupt temperature drop upon entering the slug $(T(\theta_{s-}) > T(\theta_{s+}))$, the saturation temperature would fall below the fluid temperature, indicating superheated rather than subcooled conditions. The thermal shock which prevents this situation is physically indicative of a narrow thermal boundary-layer (of thickness Pe^{-1}) which lies within the slug at its left extremity.

The second law (13*b*) admits the temperature jump at θ_s, provided that $F_h \geqslant 0$ at θ_{s+}. Using the definition of F_h and the condition that $P' =$ constant within the slug, it is seen that

$$\frac{dF_h}{d\theta} > 0.$$

It follows that $F_h > 0$ at the right end of the slug which, from second-law considerations (13*c*), rules out a temperature jump at the right. The absence of a right-hand temperature jump serves to determine the extent of the liquid slug, as explained in the numerical procedure of the appendix.

A typical imbedded-slug flow is presented in figure 3. Although a thermal shock occurs only at the left end of the slug, a saturation shock is found to occur at both ends, as in the back-to-back shocks of Fayers' hot-water flood problem. The width of the slug depends upon the given data.

(*a*) Upon decreasing $S(0)$, the slug solution properly transforms into the strictly two-phase solution of § 4. As $S(0) \to S^*(0)$ from above, the width of the slug and the jump in T both approach zero.

(*b*) An increase in $S(0)$ causes increased slug-width but only to a finite extent as $S(0) \to 1$.

(*c*) An increase in ΔT (i.e. T_0/T_∞) causes increased slug-width, because more condensate is then produced in raising the temperature of the solid matrix.

(*d*) An increase in $S(\infty)$ causes increased slug-width. As $S(\infty) \to 1$, the slug extends toward infinity, and the liquid compressibility ψ must be taken into account. For a fully-wet far field (i.e., $S(\infty) = 1$), the pressure disturbance penetrates to a relatively large depth $\theta \sim (P_0 \psi \mu_v/\mu_l)^{\frac{1}{2}}$, (roughly, $P \sim \operatorname{erfc}(x(\mu_l \epsilon \psi/Kt)^{\frac{1}{2}})$ in the far field) compared to the two-phase condensation region which remains confined to a boundary-layer of thickness $\theta \sim (\rho_v/\rho_l)^{\frac{1}{2}}$.

6. Superheated/two-phase...

Under superheated inflow conditions

$$S_0 = 0, \quad T_0 \geqslant T_{\text{sat}}(P_0),$$

there is a narrow superheated-vapour zone adjacent to the entrance, followed by a two-phase downstream region (perhaps containing an imbedded slug of liquid) like that described previously.

A shock with $[T] < 0$ must occur in passing from the superheated region into the two-phase region. To show this, first note that $F_h < 0$ at $\theta = 0$ and that F_h must then remain negative throughout the superheated region. Otherwise, there is a singularity in the energy equation (8). Now, with $F_h < 0$ and $P' < 0$,

$$T' = 0 \quad \text{and} \quad \frac{dT_{\text{sat}}}{d\theta} < 0,$$

indicating that the flow becomes more superheated as θ increases. A temperature drop must, therefore, occur in the superheated/two-phase transition.

The shock into the two-phase region occurs when $F_h = 0$. This is demonstrated by examining the shockless behaviour of the system for small (but now non-zero) values of the thermal conductivity $\langle k \rangle$ and the capillary pressure P_c which respectively appear as multipliers of T_{xx} and S_{xx}.

(*a*) In the shock-like transition region there are sharp gradients in T but not in P. Such a situation cannot occur in a two-phase region where the Clausius–Clapeyron equation relates T' and P'. Thus, the sharp gradients in T must occur in the single-phase region.

(*b*) When $\langle k \rangle$ and P_c are both non-zero, T and T' are both continuous in going from the superheated region to the two-phase. Thus, the sharp temperature gradients of the single-phase region must flatten out *before* entering the two-phase region.

(*c*) F_h must change sign (i.e., become positive) in the single-phase region. Otherwise, T' could not flatten out. This assertion is based on the extended form of the energy equation (8) which includes thermal conduction

$$Pe^{-1}T'' = -(F_h)T', \quad 0 < Pe^{-1} \ll 1.$$

Once the temperature gradient becomes negative ($T' < 0$), it grows progressively steeper ($T'' < 0$), unless F_h becomes positive ($F_h = +\delta$).

(*d*) F_h must not change sign (i.e. remains negative) throughout the single-phase region and in crossing the shock into the two-phase region. This condition is a consequence of the shock relations, as previously noted in § 3.

To resolve the apparent contradiction between (*c*) and (*d*), it is concluded that $F_h = +\delta \simeq 0$ at the superheated/two-phase transition. This conclusion rigorously satisfies the continuous boundary-layer argument (*c*) and approximately satisfied the lower-order shock-layer argument (*d*) as $\delta \to 0$. The condition that $F_h = 0$ serves to determine the position of the superheated/two-phase shock, as described in the numerical procedure of the appendix.

The typical superheated/two-phase flow of figure 4 is somewhat comparable to Morrison's numerical calculation for a condensing steam/water flow in the presence of confluent air. The superheated region is always quite small, even for large values of $T_0/T_{sat}(P_0)$. Furthermore, an increase in T_0 has very little effect on the downstream solution, as apparent in a comparison of figures 1 and 4. The effects of superheat are small because C_v is small (compared to h_{lv}), and hence the flow is desuperheated in a region which is narrow (compared to the condensation region). In taking the limit as $C_v \to 0$, the superheated region shrinks to zero, the temperature shock moves to the inlet, and the two-phase equations start off singular.

When the amount of superheat approaches zero, (i.e. $(T(0) - T_{sat}(P(0))) \to 0$), the breadth of the superheated region remains finite. This behaviour is a consequence of the energy equation (4*b*) which demands that $T'(0) = 0$ whenever $S(0) = 0$ (provided that $C_v \neq 0$). Since

$$T' = 0 \quad \text{and} \quad P' < 0 \Rightarrow \frac{dT_{sat}}{d\theta} < 0 \quad \text{at} \quad \theta = 0,$$

it is concluded that for $S(0) = 0$ the saturation temperature dives below the fluid temperature, resulting in a superheated region at the inlet. It turns out that this tendency toward superheat persists for small (but non-zero) values of $S(0)$ as described in the next section.

7. Two-phase/superheated/two-phase/...

For small values of S_0 there is a two-phase region adjacent to the boundary, followed by a superheated zone, followed by a two-phase downstream region (perhaps containing an imbedded slug of liquid). The occurrence of an imbedded superheated region can be explained on the basis of mathematical or physical arguments. The differential equations demand that, for $S(0) \neq 0$, $S'(0) \sim h_{lv}P'(0)$, as also apparent in figure 2, indicating that the flow must become dryer as it moves forward into the medium. Physically, the inflowing fluid experiences a decreasing pressure ($DP/Dt < 0$) which, according to the Clausius–Clapeyron equation, must be accompanied

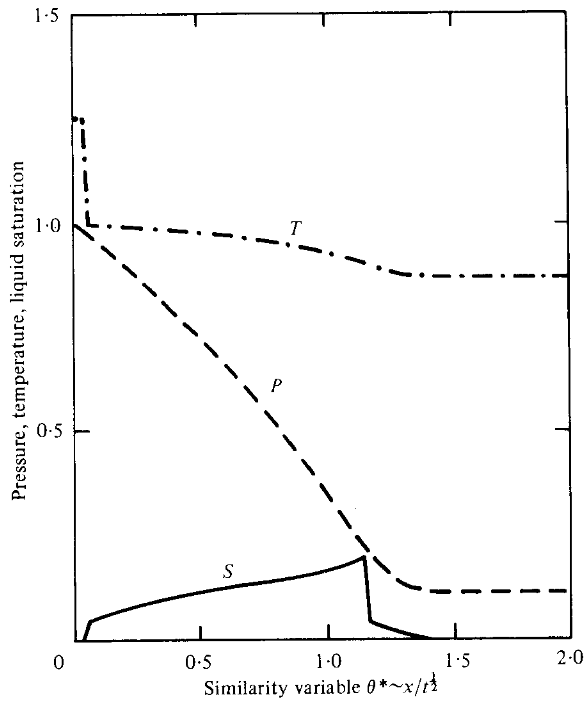

FIGURE 4. Two-phase flow with superheated inflow region (otherwise same as figure 1).

by a decreasing temperature ($DT/Dt < 0$), and this cooling is apparently accomplished by evaporation of the liquid ($DS/Dt \sim \partial S/\partial x < 0$, at the boundary where conditions are fixed in time).

Downstream of the two-phase entry, the structure is identical to the previous superheated flow. So, the only new feature is the two-phase/superheated transition. There cannot be a temperature jump in passing forward from the two-phase region into the superheated region. Supposing to the contrary that $[T] \neq 0$, the second law (13*b*) requires that $F_h \geqslant 0$. Then, from the continuity equation (9*b*) and the definition of F_h, it is seen that

$$\frac{dF_h}{d\theta} > 0$$

in the superheated region, so $F_h > 0$ at the right end of the region. This is in contradiction with the logic of the previous section which showed that $F_h \leqslant 0$ at the right end. Hence, $[T] = 0$ at the two-phase/superheated transition.

A stopping condition is needed to determine the location of the two-phase/superheated transition. Since $[T] = 0$, the shock conditions require that $[F_v] = [F_l] = 0$. Further, since $S = 0$ and $F_l = 0$ on the superheated side, it follows that

$$F_l = (\tfrac{1}{2}\theta + P'RS^2)\,S = 0 \Rightarrow \tfrac{1}{2}\theta + P'RS^2 = 0, \tag{19}$$

on the two-phase side. Here we have ruled out the possibility that $S = 0$, since this would require a singularity ($\det \mathbf{A} = 0$) in the two-phase region.

Flows which enter under two-phase conditions may have different character, depending on the value of $S(0)$. The two-phase/superheated/two-phase solution of

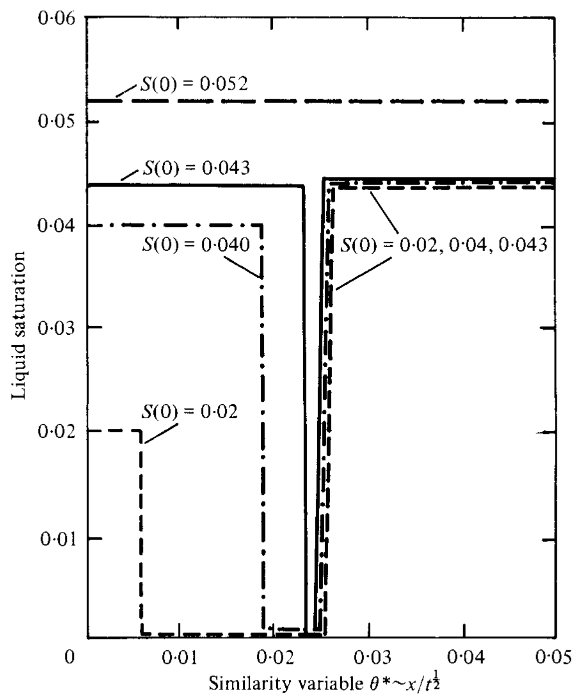

FIGURE 5. Family of saturation profiles showing imbedded superheated zone for small enough $S(0)$.

this § 7 is valid for small $S(0)$ but fails when $S(0)$ is too large. Conversely, the strictly two-phase solution of § 4 is valid for large $S(0)$ but fails when $S(0)$ is too small. To demonstrate continuous dependence on data (i.e. on $S(0)$) and the nature of the type 7/type 4 transition, computer runs were made for a succession of $S(0)$ values, starting from $S(0) = 0$, as illustrated in figure 5. Letting θ^* be the point at which the two-phase region ends and the superheated region begins, a necessary condition for the superheated region to exist is that $F_h(\theta^*) \leqslant 0$. As $S(0)$ increases, $F_h(\theta^*)$ increases until it approaches 0. At this point, the superheated region has shrunk to zero length, since it ends when $F_h = 0$. Above this value of $S(0)$, the method of § 4 is applicable.

8. Summary

Shock-like phenomena are seen to occur in transient condensing flow through porous media. A pressure-driven, phase-changing flow will develop steep gradients in saturation and temperature, approaching shock-like behaviour when the dispersive effects of capillarity and heat conduction are small. Several different flow structures may occur, depending upon the initial (i.e. far field) and boundary (i.e. inflow) data:

(*a*) *Two-phase/two-phase.* Strictly two-phase flows occur when: $S(0)$ is not too small; $S(\infty)$ is not too large; and ΔT is not too large. A two-phase/two-phase saturation shock divides the flow, but thermal shock is absent.

(*b*) *Two-phase/subcooled liquid/two-phase.* For large $S(0)$, large $S(\infty)$, or large ΔT, the two-phase/two-phase shock line broadens into a subcooled liquid region of finite width. Saturation shock occurs at both ends of the liquid slug, accompanied by thermal shock on the trailing end.

(*c*) *Two-phase/subcooled liquid.* As $S(\infty) \rightarrow 1$, the subcooled region extends to infinity and the compressibility of the liquid must be taken into account.

(*d*) *Two-phase/superheated/two-phase/....* For small $S(0)$, an imbedded superheated region appears near the boundary. Saturation shock occurs at both ends of the superheated zone accompanied by thermal shock on the leading end. As $S(0) \rightarrow 0$, the superheated zone extends backward to the entrance and the left-hand shock shrinks to zero leaving only a superheated/two-phase/...structure.

(*e*) *Superheated/two-phase/....* With $S(0) = 0$, the inflow may be superheated, causing accentuation of the superheated region. However, the width of the superheated zone depends strongly on the specific heat ratio (vapour to solid), not on the amount of superheat. As $C_V \rightarrow 0$, the superheated zone collapses into the origin, leaving a thermal-shock and a singularity ($\det \mathbf{A} = 0$) at the origin.

In all cases, the transitions from one flow-structure into another depend continuously on the data. There are many possible combinations of inflow and far-field structure, for example, two-phase/superheated/two-phase/subcooled/two-phase.

Further study of the condensing flow problem is reported in another paper (Nilson & Romero 1980) where we restrict to a representative case in which the inflow and far-field are both prescribed as dry saturated-vapour states. Particular emphasis is given to the various length scales which arise in the phase-change flows. The overall penetration depth of the flow is a consequence of the gross energy-balance and momentum-balance, as embodied in the scaling of the similarity variable (used in the present figures),

$$\theta^* = \frac{x}{t^{\frac{1}{2}}}\left(\frac{\epsilon\mu_V}{K\Delta P}\right)^{\frac{1}{2}}\left(\frac{\Delta S\rho_{lv}}{\rho_{v0}}\right)^{\frac{1}{2}}$$

in which

$$\Delta S = \langle\rho C\rangle_0 \Delta T/\epsilon\rho_l(h_{lv})_0 \quad \text{and} \quad \langle\rho C\rangle_0 = (1-\epsilon)\rho_m C_m + \epsilon\Delta S\rho_l C_l.$$

But, there are several boundary-layer zones within the flow field:

1. The imbedded superheated zone (as in figure 5) collapses into a singularity at $\theta = 0$, as $C_v\Delta T/h_{lv} \rightarrow 0$.

2. The precursor two-phase zone, which lies ahead of the subcooled-liquid zone, vanishes as $(\Delta S\,\rho_{lv}/\rho_{v0}) \rightarrow \infty$.

3. The thermal boundary-layer, which lies at the trailing edge of the subcooled-liquid zone, approaches a thermal shock as $Pe \rightarrow \infty$.

4. The increasing saturation region ($S' > 0$), which lies to the left of all shocks, collapses into the origin as $R = (\mu_v/\mu_l) \rightarrow 0$. The central shock front moves backward into the origin in this singular immobile-liquid limit.

In the companion paper (Nilson & Romero 1980) discussion of these matters is illustrated by a family of calculations concerning steam-flow in sandstone.

A complex flow structure has been encountered, even in a rather elementary single-component, one-dimensional problem. It is not, however, suggested that a detailed knowledge of the flow structure is always a critical issue in the analysis of engineering systems, particularly in the geologic applications where there is only limited knowledge concerning the structure of the porous medium. Nevertheless, it is important to understand the fine structure which is predicted by the customary and well-established mathematical model. It is only through this knowledge that the appropriate engineering approximations and computational tools can be formulated and tested.

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Appendix

The numerical integration procedure is based upon the well-known shooting method. Standard library routines perform the major operations: integration by a fifth-order Runga–Kutta method and iterative adjustment of the shooting parameters by a simplex minimization procedure. A general outline which includes all of the special cases is as follows:

(1) Guess the values of $P'(0)$ and θ_s.

(2) Integrate the two-phase equations (6) until $F_l = 0$.

(3) Shock according to (10) with $S = 0$ on the right.

(4) Integrate the superheated equations (8) and (9*b*) until $F_h = 0$.

(5) Shock according to (14*a*, *b*) with $T = T_{sat}(P)$ on the right. Here (14*a*, *b*) replaces (10) because $F_h = 0$ in (10*b*).

(6) Integrate the two-phase equations (6) until $\theta = \theta_s$.

(7) Shock according to (10) with $S = 1$ on the right.

(8) Integrate the subcooled liquid equations (8) and (9*a*) until $T = T_{sat}(P)$.

(9) Shock according to (14*a*, *b*) with $T = T_{sat}(P)$ on the right.

(10) Integrate the two-phase equations (6) out to large θ.

A minimization procedure adjusts the values of the shooting parameters, $P'(0)$ and θ_s, until both of the far-field boundary conditions are satisfied.

Although it is possible that all of the integration steps might apply to a particular flow, there are also a number of subset procedures which generate the simpler flows that are described in the individual sections of the paper.

§4.	Two-phase/two-phase	(1), (6), (9)–(10)
§5.	Two-phase/subcooled/two-phase	(1), (6)–(10)
	Two-phase/subcooled	(1), (6)–(8)
§6.	Superheated/two-phase...	(1), (4)–(7), (8), (9), (10)
§7.	Two-phase/superheated...	(1)–(7), (8), (9), (10)

The presence of the subcooled slug, (7) and (8), is optional in §§ 6 and 7.

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